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Engineering Notes

Using Edelbaum's Method to Compute Low-Thrust Transfers with Earth-Shadow Eclipses

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Introduction

IN HIS seminal paper, Edelbaum developed analytical solutions for transfers between inclined circular Earth orbits [1], and these results serve as an excellent preliminary design tool for estimating the velocity increment ΔV and transfer time for low-thrust missions with continuous thrust. Real solar electric propulsion (SEP) spacecraft, however, experience periods of zero thrust during passage through the Earth's shadow, and this major effect is not accommodated in Edelbaum's analysis. Colasurdo and Casalino [2] have extended Edelbaum's analysis [1] and developed an approximate analytic technique for computing optimal quasi-circular transfers with the inclusion of the Earth's shadow. Only coplanar transfers are considered, and the thrust steering is constrained so that the orbit remains circular in the presence of the Earth's shadow. Kechichian [3] also developed an analytical method for obtaining coplanar orbit-raising maneuvers in the presence of Earth shadow, where eccentricity is constrained to remain zero. Both [2,3] develop suboptimal solutions for the coplanar circle-to-circle transfer problem with Earth-shadow arcs, since steering the thrust vector to maintain zero eccentricity ultimately leads to steering losses compared with the minimum-time transfer.

Low-thrust trajectory optimization programs [4–6] have been developed to obtain optimal transfers in the presence of Earth-shadow eclipses, but these techniques can suffer from the usual pitfalls associated with numerical search algorithms (uncertain convergence, high computational loads, slow run times, etc.). Edelbaum's continuous-thrust solution [1] is attractive because it is analytic and requires no numerical integration of the system differential equations. A low-thrust trajectory program based on analytic methods that can accurately accommodate Earth-shadow arcs would be a useful mission design tool.

This Note presents a new low-thrust trajectory program that is based on Edelbaum's analytic solution [1]. This new algorithm can accurately compute the transfer time for low-thrust maneuvers in the presence of Earth-shadow arcs without relying on numerical integration of the equations of motion. Several cases comparing the performance of the new algorithm with numerically integrated optimal trajectories are presented in order to demonstrate its effectiveness.

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Kechichian's Algorithm for Edelbaum's Analytic Solution

Edelbaum's original analysis involved a low-thrust transfer between two circular orbits with a prescribed plane change [1]. The major assumptions of Edelbaum's work are 1) propulsive thrust is continuous during the transfer, 2) thrust acceleration is constant during the transfer, 3) the transfer is quasi circular (eccentricity remains zero), and 4) the magnitude of the out-of-plane (yaw) steering angle is held constant during an orbital revolution. Edelbaum derived analytic expressions for the total ΔV and transfer time t_f. Kechichian [7] reformulated Edelbaum's problem [1] by applying optimal control theory to the minimum-time transfer problem and, consequently, derived analytic expressions for the time histories of semimajor axis, inclination, yaw angle, and ΔV. The proposed technique for computing low-thrust transfers in the presence of Earth-shadow eclipses is based on Kechichian's algorithm, which is summarized as follows (see [7] for details).

The three-dimensional circle-to-circle orbit-transfer problem is defined by the initial semimajor axis a_0, the final semimajor axis a_f, the desired inclination change $\Delta i = |i_f - i_0|$, and the constant thrust acceleration $f = T/m_0$, where T is the thrust and m_0 is the initial spacecraft mass. Initial and final circular velocities, V_0 and V_f, are computed easily from the respective radii (or semimajor axes). Kechichian's algorithm [7] begins by calculating the magnitude of the initial yaw-steering angle β_0:

$$\tan\beta_0 = \frac{\sin[(\pi/2)\Delta i]}{(V_0/V_f) - \cos[(\pi/2)\Delta i]} \tag{1}$$

Next, the total ΔV is computed using

$$\Delta V_{\text{total}} = V_0\cos\beta_0 - \frac{V_0\sin\beta_0}{\tan[(\pi/2)\Delta i + \beta_0]} \tag{2}$$

which yields the same result as Edelbaum's analytic expression [1] for total ΔV:

$$\Delta V_{\text{total}} = \sqrt{V_0^2 + V_f^2 - 2V_0V_f\cos[(\pi/2)\Delta i]} \tag{3}$$

Finally, the transfer time using continuous thrust is computed simply from the total velocity increment and constant thrust acceleration:

$$t_f = \frac{\Delta V_{\text{total}}}{f} \tag{4}$$

Kechichian's algorithm [7] determines the time histories of the important state and control variables for the time interval $0 \le t \le t_f$. The velocity increment is a linear function of time:

$$\Delta V(t) = ft \tag{5}$$

The semimajor axis during the transfer is computed from the circular orbital velocity:

$$a(t) = \frac{\mu}{V_0^2 + f^2t^2 - 2V_0ft\cos\beta_0} \tag{6}$$

where μ is the Earth's gravitational parameter. Inclination is calculated using

$$i(t) = i_0 + \operatorname{sgn}(i_f - i_0)\frac{2}{\pi}\left[\tan^{-1}\left(\frac{ft - V_0\cos\beta_0}{V_0\sin\beta_0}\right) + \frac{\pi}{2} - \beta_0\right] \tag{7}$$

The signum function is required, because the desired inclination change can be either positive or negative, and Kechichian's formulation [7] sets the yaw angle magnitude β_0 as positive. The time history of the yaw angle $\beta(t)$ can also be computed using Kechichian's method, but it is not needed. In summary, analytic expressions for semimajor axis $a(t)$, inclination $i(t)$, and velocity increment $\Delta V(t)$ are determined easily from Kechichian's reformulation of Edelbaum's [1] three-dimensional quasi-circular low-thrust transfer problem.

Low-Thrust Transfers with Earth-Shadow Eclipses

The primary goal is to use the analytic Edelbaum [1] (via Kechichian [7]) solution and incorporate the effects of discontinuous thrust caused by Earth-shadow eclipses. Clearly, a given orbital transfer with interrupted thrust will require more time when compared with the continuous-thrust case; therefore, the continuous- and discontinuous-thrust state trajectories will exhibit different time scales. However, the state trajectories with energy (semimajor axis) as the independent variable exhibit similar profiles for both the continuous- and discontinuous-thrust cases, as demonstrated by the following example.

Velocity Increment Versus Semimajor Axis

Figure 1 shows ΔV vs semimajor axis for an orbit transfer between circular low Earth orbit (LEO) and geostationary equatorial orbit (GEO). The initial LEO has $a_0 = 6928$ km and inclination $i_0 =$ 28.5 deg, while the target GEO has $a_f = 42,164$ km and $i_f = 0$. The solid (top) curve is the Edelbaum [1] ΔV vs a, computed using Eqs. (5) and (6) for constant thrust acceleration $f = 0.3348$ mm/s^2. However, the Edelbaum ΔV vs a curve, shown in Fig. 1, is valid for any value of f; the Edelbaum $\Delta V(a)$ history solely depends on the initial and target orbits.

Three minimum-time LEO–GEO transfers that include Earth-shadow eclipse effects are obtained using a direct optimization method and sequential quadratic programming [6,8]. This method propagates the trajectory through numerical integration of the Gaussian form of Lagrange's planetary equations. Orbital-averaging techniques are employed in order to allow relatively large integration steps (on the order of days), and the optimal pitch and yaw thrust-steering profiles are determined by direct optimization of the costate multipliers from optimal control theory. Earth-shadow and oblateness (J_2) effects are included in the trajectory propagation. Shadow exit and entrance angles are computed from the intersection of the instantaneous orbital plane and a cylindrical shadow model [9]. Three values of initial thrust acceleration were used ($f_0 = 0.3348$, 0.1674, and 0.9206 mm/s^2) with corresponding mass-flow rates of 0.0124, 0.0062, and 0.0939 g/s. This wide range in thrust acceleration and mass-flow rate resulted in minimum transfer times ranging from 68 to 404 days. However, despite the inclusion of Earth-shadow effects and the wide range in f and mass-flow rate, Fig. 1 shows that the optimal $\Delta V(a)$ profiles nearly match the Edelbaum solution [1]. It is interesting to note that total ΔV from the optimal trajectories is slightly less than Edelbaum's total ΔV (average optimal $\Delta V = 5.6839$ km/s, Edelbaum $\Delta V =$ 5.8200 km/s). The optimized trajectories exhibit lower ΔV compared with the Edelbaum solution, because yaw angle magnitude is allowed to modulate over an orbital revolution (β is held constant for each revolution in Edelbaum's solution, switching signs at the antinode crossings). For the optimal transfers, the out-of-plane thrust component is not wasted near the antinode crossings where $\mathrm{d}i/\mathrm{d}t = 0$, regardless of the yaw angle. Furthermore, the optimal transfers exhibit a slight delay in the inclination change when compared with the Edelbaum inclination profile; hence, a greater portion of the plane change is performed at higher altitudes, which improves the ΔV performance. The numerically integrated optimal transfer does incur some steering losses due to pitch-steering maneuvers required to reduce the inevitable buildup of eccentricity caused by the Earth-shadow effect (Edelbaum's solution assumes tangent steering for the in-plane thrust component to maximize the rate of energy gain). However, these losses are small and less than the performance gain achieved by optimal modulation of the yaw steering angle over an orbital revolution.

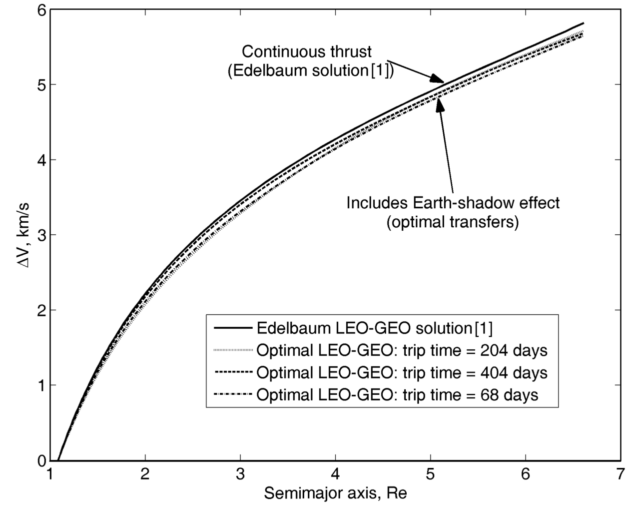

Fig. 1 Velocity increment vs semimajor axis for LEO–GEO transfers.

Figure 1 shows that $\Delta V(a)$ is determined by the orbit-transfer parameters a_0, a_f, and Δi and exhibits very little dependence on vehicle parameters (such as thrust, mass, and mass-flow rate) or the inclusion of Earth eclipses. Therefore, it appears that Edelbaum's analytic $\Delta V(a)$ profile [1] can be used to estimate the transfer time when Earth-shadow eclipses are present.

Transfer Time with Discontinuous Thrust

To begin the analysis of transfer time, rewrite Eq. (5) as a difference equation for propagating the velocity increment ahead in time:

$$\Delta V_{k+1} = \Delta V_k + f\Delta t; \qquad k = 0, 1, 2, \ldots, N \tag{8}$$

where $\Delta t = t_{k+1} - t_k$. For the discontinuous-thrust case, it is assumed that $\Delta V(a)$ essentially matches the Edelbaum solution [1] for the desired orbit maneuver (as shown in Fig. 1); therefore, ΔV_k is determined by evaluating the Edelbaum solution at discrete intervals. Therefore, Eq. (8) can be used to obtain the time step required for a known increase in ΔV:

$$t_{k+1} = t_k + \frac{(\Delta V_{k+1} - \Delta V_k)}{f}; \qquad k = 0, 1, 2, \ldots, N \tag{9}$$

However, Eq. (9) must be modified, since thrust acceleration $f = T/m$ is not constant for two reasons: 1) spacecraft mass m decreases during the transfer and 2) thrust T is discontinuous due to the Earth's shadow. A modified version of Eq. (9) is

$$t_{k+1} = t_k + \frac{(\Delta V_{k+1} - \Delta V_k)}{\bar{f}_k w_k} \tag{10}$$

The transfer time required to go from step k to step $k+1$ is the corresponding increment in Edelbaum ΔV [1] divided by the product of the average thrust acceleration $\bar{f}_k$ and a weighting function w_k. The average thrust acceleration $\bar{f}_k$ is computed by dividing the thrust magnitude (T, assumed to be constant) by the average mass between discrete steps k and $k+1$. Spacecraft mass can be computed from the rocket equation and specific impulse I_{sp}, since ΔV is known. The weighting function w_k represents the percentage of time the spacecraft is thrusting during one revolution. For example, $w_k = 1$ represents the continuous-thrust case where the spacecraft is entirely in sunlight during one revolution (no Earth shadow); if $w_k = 0.75$, then the spacecraft is in sunlight for three quarters of an orbital revolution. The weighting function is computed from the Earth-shadow angle, $\Delta\theta_{SH}$:

$$w = 1 - \frac{\Delta\theta_{SH}}{2\pi} \quad (11)$$

The shadow arc $\Delta\theta_{SH}$ is computed by using Neta and Vallado's algorithm [9], which requires the osculating orbit's size, shape, and orientation, and the current date in order to establish the Earth–sun vector in the geocentric-equatorial frame. Because Edelbaum's solution is used to determine osculating elements a and i, it is assumed that the orbit transfer is quasi circular; therefore, eccentricity remains zero, and argument of perigee is arbitrarily set to zero. For a real SEP transfer, interrupted thrust will tend to initially increase eccentricity (the apogee will lie in the Earth's shadow); however, numerically integrated optimal transfers show that the peak eccentricity never exceeds 0.15 and is often no more than 0.1. It should be emphasized that using Edelbaum's quasi-circular solution [1], while it does not accurately predict the eccentricity history, provides a simple, analytic method for computing the osculating orbital elements that are required for shadow-arc calculations. The variation in eccentricity that occurs in a numerically integrated SEP trajectory will have a small effect on shadow-arc computations when compared with the quasi-circular case (that is, accurate prediction of a, i, and ascending node angle Ω have the greatest effect on computing the Earth-shadow entrance and exit angles when eccentricity variation is small). Finally, it should be noted that the proposed method does not rely on numerical integration of the differential equations governing elements a, i, or e (nor does it modulate the thrust-steering direction); hence, there is no mechanism for controlling eccentricity.

Ascending node angle and inclination are required to establish orbital-plane orientation for the Earth-shadow algorithm. Out-of-plane (yaw) thrust does not contribute to $\mathrm{d}\Omega/\mathrm{d}t$, since yaw steering amplitude switches sign at antinode crossings. Earth's oblateness J_2 has the greatest effect on the line of nodes, and the averaged rate for ascending node due to J_2 is

$$\dot{\Omega} = \frac{-3}{2} J_2 n \left(\frac{R_E}{a}\right)^2 \cos i \quad (12)$$

where R_E is the Earth's equatorial radius, $J_2 = 1.0826269(10^{-3})$, and $n = \sqrt{\mu/a^3}$ is the mean orbital motion. The ascending node angle is propagated ahead in time by using the simple first-order equation,

$$\Omega_{k+1} = \Omega_k + \dot{\Omega}_k \Delta t \quad (13)$$

The transfer-time calculation for quasi-circular transfers in the presence of Earth shadow can now be summarized: given the initial and target circular orbit radii, inclination change, and initial thrust acceleration f_0, compute the total ΔV and transfer time t_f for Edelbaum's solution [1] for the continuous-thrust case using Eqs. (2) and (4). Divide the continuous-transfer time t_f into N segments, where the discrete histories for ΔV_k, a_k, and i_k are computed using Eqs. (5–7). Next, compute the Earth-shadow arc $\Delta\theta_{SH}$ with knowledge of the departure date and the initial orbital elements (including Ω_0) and compute the weighting function w_k using Eq. (11). Use Eq. (10) to compute the increment in transfer time that accounts for the Earth-shadow effect by employing the average thrust acceleration $\bar{f}_k$ and the weighting function w_k. Finally, propagate the ascending node angle ahead in time using Eqs. (12) and (13). The recursive equations are repeated until all N segments of Edelbaum's solution have been processed.

One additional note is required. Figure 1 shows that all three numerically integrated, optimized LEO–GEO transfers exhibit slightly lower total ΔV when compared with Edelbaum's analytic result (2) [1], for reasons previously explained. Therefore, in order to accurately predict the optimal transfer time in the presence of Earth shadow, the Edelbaum ΔV in Eq. (10) is scaled by an optimization factor c, where $c \in [0, 1]$. Several numerically optimized SEP transfers were obtained for a range of circular orbits, thrust accelerations, and initial ascending node angles, and the average ratio between the optimized ΔV and Edelbaum's ΔV was found to be 0.98; therefore, the optimization factor was fixed at $c = 0.98$.

Numerical Results

The utility of the proposed algorithm is demonstrated by two orbit-transfer scenarios: 1) LEO–GEO transfer and 2) LEO to mid-Earth orbit (MEO) transfer. In both cases, initial circular LEO has $a_0 = 6928$ km with inclination $i_0 = 28.5$ deg. GEO boundary conditions are $a_f = 42,164$ km and $i_f = 0$ deg. The MEO target represents a 12 h Global Positioning System (GPS) orbit with $a_f = 26,578$ km and an inclination of $i_f = 55$ deg.

Several LEO–GEO transfers are obtained using the Edelbaum-based method for a near-term ion-propulsion spacecraft with initial mass $m_0 = 1,200$ kg, $I_{sp} = 3,300$ s, input power $P =$ 10 kW, and thruster efficiency $\eta = 65\%$. The Edelbaum solution is divided into 100 segments ($N = 100$ in the recursive computations), and the real-time run time of the proposed method on a Pentium M laptop is less than 0.1 s for a single-trajectory solution. The LEO departure date is fixed at 21 March 2000 for shadow calculations. Figure 2 shows the LEO–GEO transfer time as predicted by the Edelbaum-based algorithm for an initial ascending node angle $0 \le \Omega_0 \le 360$ deg. Several minimum-time LEO–GEO transfers are obtained using the direct optimization method [6,8] for a variety of initial ascending node angles, and the corresponding transfer times are shown in Fig. 2 by the discrete symbols. Varying the initial ascending node angle causes different Earth-shadow histories that, in turn, effect the total transfer times. Figure 2 shows that the Edelbaum-based solutions exhibit a good match with the numerically optimized transfers, and the Edelbaum-based solutions accurately predict the trend in transfer time as Ω_0 varies. The best (shortest) LEO–GEO transfer times are 201.86 days (optimal) and 201.94 days (Edelbaum based); the worst (longest) transfer times are 213.93 days (optimal) and 213.71 days (Edelbaum based). The average of the transfer-time errors between the Edelbaum-based and optimal solutions is 0.81% (or about 1.7 days).

Several LEO–MEO (GPS orbit) transfers are obtained using the Edelbaum-based method for a spacecraft equipped with near-term Hall-effect thrusters with $I_{sp} = 1,600$ s, input power $P = 10$ kW, and thruster efficiency $\eta = 45\%$. Initial spacecraft mass in LEO is 1200 kg, and the LEO departure date is fixed at 21 March 2000. Figure 2 also shows that the LEO–GPS transfer times as predicted by the Edelbaum-based method exhibit a very good match with the numerically optimized transfer times. The best (shortest) LEO–GPS transfer times are 118.52 days (optimal) and 119.30 days (Edelbaum based); the worst (longest) transfer times are 132.50 days (optimal) and 131.69 days (Edelbaum based). The average of the transfer-time errors between the Edelbaum-based and optimal solutions is 0.68% (or about 0.9 days).

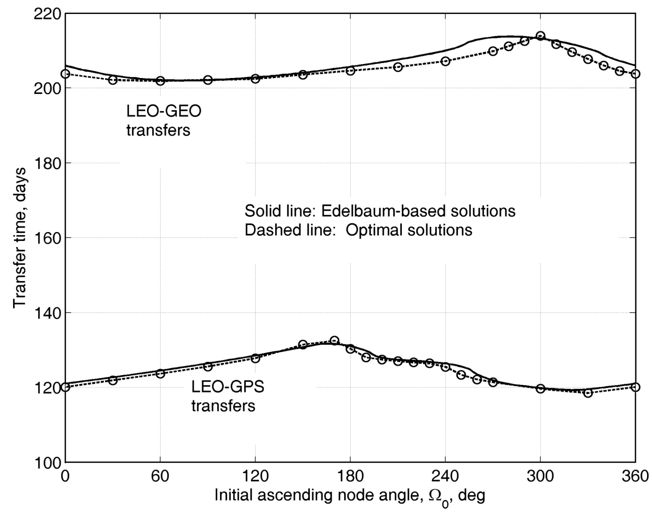

Fig. 2 Transfer time vs initial ascending node angle.

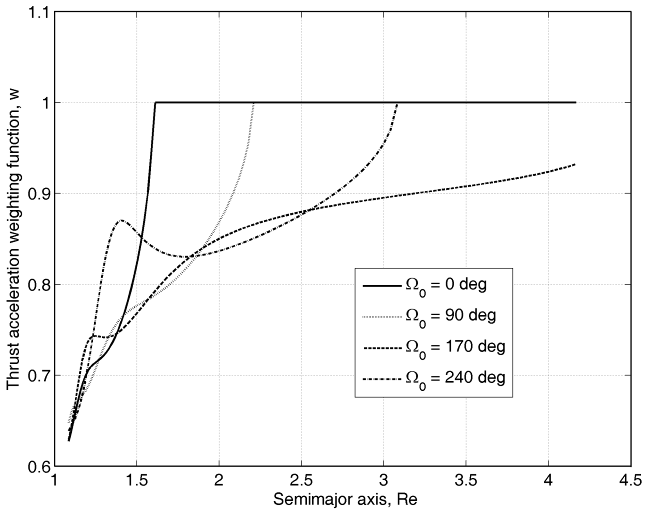

Fig. 3 Weighting function *w* vs semimajor axis for LEO–GPS transfers.

Figure 3 shows the thrust acceleration weighting function w plotted against semimajor axis a for LEO–GPS transfers with four different initial ascending node angles. All four orbit transfers begin with the spacecraft in sunlight roughly 63–65% of the time in LEO, but the case with $\Omega_0 = 0$ deg reaches an orbit-sun geometry at $a = 1.61\ Re$, such that the spacecraft is always in sunlight for the remainder of the transfer. Consequently, the transfer time for this case is the shortest of the four, as verified by Fig. 2. The weighting function for the case with $\Omega_0 = 170$ deg never reaches unity; therefore, the spacecraft always experiences an eclipse during every revolution and, hence, takes the longest time to reach its target, as shown in Fig. 2.

Conclusions

A new semianalytic algorithm has been developed for computing low-thrust orbit transfers in the presence of Earth-shadow eclipses. The new method uses Kechichian's reformulation [7] of Edelbaum's analytic trajectory solution [1]. Edelbaum's method is used to analytically compute the histories for ΔV, semimajor axis, and inclination. The key feature is using the Edelbaum-based orbital elements to compute the history of the Earth-shadow arc during the orbit transfer that, in turn, is used in a weighting function applied to the thrust acceleration. Transfer time in the presence of Earth shadow is ultimately computed from the Edelbaum ΔV and weighted thrust acceleration. The resulting algorithm is extremely fast and does not require numerical integration of equations of motion or numerical iteration for convergence. Many solutions are obtained for two orbit-transfer cases between inclined circular orbits, and the transfer times predicted by the Edelbaum-based method show an excellent match with the numerically optimized trajectories. These results indicate that the new algorithm would serve as a valuable preliminary design tool for solar electric propulsion spacecraft missions.

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